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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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that the integral

$$\int_{\beta}^{\infty(O_1)} \phi(z)\, e^{-sf(z)}\, dz = \int_{(\beta M')} + \int_{(M'N')} + \int_{N'}^{\infty} \phi(z)\, e^{-sf(z)}\, dz,$$

$$= \text{an integral function of } s + \int_{\rho_0}^{\infty} \phi(x)\, e^{-sf(x)}\, dx,$$

so that our formula (15.2) becomes

$$\text{(15.3)} \qquad H(s) = \bar{G}(s) + \int_{\rho_0}^{\infty} \phi(x)\, e^{-sf(x)}\, dx,$$

where $\bar{G}$ is some integral function of s.

Similarly, we may consider other theorems under the hypotheses made at the beginning of this section.

16. Now we are in a position to consider the question raised at the end of § 5, namely, whether we can assign a meaning to such a divergent series as

$$\text{(16.1)} \qquad \sum_{1}^{\infty} e^{Ain^{\gamma} + kn^{\beta} - sn^{\alpha}}, \qquad k > 0, \qquad 1 \geqslant \gamma > \beta > \alpha > 0,$$

$$(A \neq 0, \text{ when } \gamma < 1;\ A \neq \text{a multiple of } 2\pi, \text{ when } \gamma = 1).$$

Euler defined the sum of the oscillatory series $\Sigma\, a_n$ as the limit, if it exists, of $\Sigma\, a_n x^n$, as $x \rightarrow 1$. It is natural to generalize this definition by saying that the sum of the series is

$$\lim_{x \rightarrow 0} \sum a_n\, e^{-x\lambda_n},$$

if it exists, where $\lambda_n \rightarrow \infty$.

Now adopting this method, we shall say that the divergent series (16.1) shall be defined by the limit of

$$H(s, z) = \sum_{1}^{\infty} \exp.\{ Ain^{\gamma} - (z - k)\, n^{\beta} - sn^{\alpha} \}, \text{ if it exists,}$$

as $z \rightarrow 0$.

Now following the lines of proof of Theorem XII, we can prove that

$$H(s, z) = G(s, z) + J(s, z),$$

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where G (s, z) is an integral function of both s and z, and the function

$$J(s, z) = \int_1^\infty \exp.\{ Aix^\gamma - (z-k)x^\beta - sx^\alpha \} \, dx.$$

Now changing the path of integration, it is easy to prove that the function J (s, z) is also an integral function of both s and z. So that

$$\lim H(s, z),$$

as $z \to 0$, exists, and is an integral function of s.

Our divergent series (16.1) is, therefore, uniquely defined by an integral function of s.

Similarly, the following divergent series

(16.2) $$\sum_1 \exp.\{ -s(\log n)^\alpha + k(\log n)^\beta + Ai(\log n)^\gamma \},$$

$$(k > 0, \ \gamma > \beta > \alpha \geqslant 1, \ A \neq 0);$$

(16.3) $$\sum_1^\infty \exp.\{ -s(\log n)^\alpha + kn^\beta + Ain^\gamma \},$$

$$(k > 0, \ \alpha \geqslant 1, \ 1 \geqslant \gamma > \beta > 0,$$

$A \neq 0$, when $\gamma < 1$, and $A \neq$ a multiple of 2π, when $\gamma = 1$); are uniquely defined by means of integral functions of s

It should be observed that none of these series is summable (C), *i.e.*, by Cesaro's means, for any value of s. For, if a series Σa_n is summable by Cesaro's means of order k, then*

$$\frac{a_n}{n^k} \to 0;$$

and it is obvious that this condition is not satisfied in any one of the above cases.

17. As an application of Theorem VIII, let us consider the series

(17.1) $$H(s) = \sum_3^\infty \phi(n)(n \log n)^{-s},$$

where $$\phi(x) = x^\alpha (\log x)^\beta \sum_0^x \frac{c_\nu}{x^\nu}, \quad \begin{cases} x \geqslant R \geqslant 3, \\ \alpha, \beta \text{ real and} \\ \beta \geqslant \alpha + \delta > \alpha, \ \delta > 0, \end{cases}$$

---

* S. Chapman, *loc. cit.*

and the series $\sum_{0}^{\infty} \frac{c_\nu}{R^\nu}$ is absolutely convergent. For $R \geqslant x \geqslant 3$,

$\phi(x)$ is of such a form that Theorem VIII is applicable.

Then, by Theorem VIII, we have

$$(17.2) \qquad H(s) = \sum_{8}^{\infty} \phi(n)\,(n \log n)^{-s}$$

$$= \int_3^R \phi(x)(x \log x)^{-s}\,dx + \int_R^\infty x^\alpha (\log x)^\beta \left(\sum_0^\infty \frac{c_\nu}{x^\nu} \right) (x \log x)^{-s}\,dx$$

$+\ G(s)$, where $G(s)$ is an integral function of s.

The first integral on the right side of (17.2) obviously represents an ntegral function of s, so that we have

$$(17.3) \qquad H(s) = G_1(s) + \int_R^\infty x^{(\alpha-s)} (\log x)^{(\beta-s)} \left(\sum_0^8 \frac{c_\nu}{x^\nu} \right) dx,$$

where G_1 is some integral function of s.

Now we have to study

$$J(s) = \int_R^\infty x^{(\alpha-s)} (\log x)^{(\beta-s)} \left(\sum_0^\infty \frac{c_\nu}{x^\nu} \right) dx,$$

which converges uniformly and absolutely in the half-plane

$$\sigma \geqslant 1 + \alpha + \varepsilon > 1 + \alpha, \ (\varepsilon > 0).$$

Also $$\int_R^\infty | x^{(\alpha-s)} | \; | (\log x)^{(\beta-s)} | \; \sum_0^\infty \left| \frac{c_\nu}{x^\nu} \right| dx$$

is convergent if $\qquad \sigma > 1 + \alpha.$

Therefore,

$$(17.4) \qquad J(s) = \sum_0^\infty c_\nu \int_R^\infty x^{(\alpha-\nu-s)} (\log x)^{(\beta-s)}\,dx,$$

$$= \sum_{0}^{N} c_\nu \int_R^\infty + \sum_{N+1}^{\infty} c_\nu \int_R^\infty .$$

(N, an arbitrary positive integer),

$$= \sum_{0}^{N} c_\nu \int_{\log R}^\infty e^{(\alpha+1-s-\nu)y}\, y^{(\beta-s)}\, dy + \sum_{N+1}^{\infty} c_\nu \int_{\log}^\infty$$

Let

$$(17.5) \qquad J_1(s) = \sum_{0}^{N} c_\nu \int_{\log R}^\infty e^{-(s+\nu-\alpha-1)\,y}\, y^{(\beta-s)}\, dy,$$

and suppose, for a moment, that s is real and $1+\beta > s > 1+\alpha$. This is possible because $\beta \geqslant \alpha+\delta > \alpha$.

Then
$$J_1(s) = \sum_{0}^{N} c_\nu \left\{ \frac{\Gamma(1+\beta-s)}{(s+\nu-\alpha-1)^{1+\beta-s}} - \int_0^{\log R} e^{-(s+\nu-\alpha-1)\,y}\, y^{(\beta-s)}\, dy \right\}.$$

In the last integral, we can expand $e^{-(s+\nu-\alpha-1)\,y}$ and integrate term by term. We shall then obtain

$$(17.6) \qquad J_1(s) = \sum_{0}^{N} c_\nu \left\{ \frac{\Gamma(1+\beta-s)}{(s+\nu-\alpha-1)^{(1+\beta-s)}} - \sum_{n=0}^{\infty} \frac{(\alpha+1-\nu-s)^n}{n!} \frac{(\log R)^{(n+1+\beta-s)}}{(n+1+\beta-s)} \right\}.$$

Now the right-side of (17.6) is an analytic function of s with the exception of the singularities given by

$$s = \alpha + 1 - \nu, \quad \nu = 0,\ 1, 2, \ldots N,$$

where N is an arbitrary positive integer. The only other singularities that can come in are those given by the equation

$$n + 1 + \beta - s = 0,$$

i.e., when $1+\beta-s = 0$, or a negative integer. But then it can be

shown easily that each such singularity of the series is neutralized by that of the first term.

Next, consider

$$(17.7) \qquad J_2(s) = \sum_{\nu=N+1}^{\infty} c_\nu \int_{\log R}^{\infty} e^{-(\nu+s-\alpha-1)y} y^{(\beta-s)} dy.$$

We want to show that this series is uniformly convergent in any finite part of the half-plane $\sigma \geqslant \alpha + \frac{1}{2} - N$.

Now $\left| \int_{\log R}^{\infty} e^{-(\nu+s-\alpha-1)y} y^{(\beta-s)} dy \right|$

$$\leqslant \int_{\log R}^{\infty} e^{-(\nu+\sigma-\alpha-1)y} y^{(\beta-\sigma)} dy$$

$$< e^{-(\nu-N-1)\log R} \int_{(\log R)}^{\infty} e^{-\frac{1}{2}y} y^{(\beta-\sigma)} dy = \frac{K}{R^\nu},$$

where K is some constant depending upon N, since the integral

$$\int_{(\log R)}^{\infty} e^{-\frac{1}{2}y} y^{(\beta-\sigma)} dy = O(1),$$

if σ is bounded.

And the series $\sum_{N+1}^{\infty} \frac{c_\nu}{R^\nu}$ converges absolutely by hypothesis. Therefore $J_2(s)$ is an analytic function of s in any finite part of the half-plane

$$\sigma \geqslant \alpha + \tfrac{1}{2} - N.$$

Taking account of the conclusions about $J_1(s)$ and $J_2(s)$, we infer that the function $J(s)$ (and hence $H(s)$) has singularities at $s = \alpha + 1 - \nu$, $\nu = c, 1, 2, \ldots$ Now if we mkke an infinite cut $((\alpha+1), -\infty)$, and interpret

$$(-\alpha - 1 + \nu + s)^{-(1+\beta-s)}$$

to mean exp. $[-(1+\beta-s)\log(s+\nu-\alpha-1)]$, where the logarithm has its principal value, then the function $J(s)$ becomes regular and uniform in this cut-plane, leaving, of course, the upper and lower edges of the cut.

The right-side of (17.6) indicates the behaviour of the function near a singularity.

18. In this section, I give an example of a Dirichlet's series to which Theorem VIII is applicable but not Theorem IX, but which, nevetheless, represents an integral function of s.

Let $$H(s) = \sum_{1}^{\infty} e^{Ai(\log n)^{\beta} - s(\log n)^{\alpha}},$$

where $A \neq 0$, α and β are real, and $\beta > \alpha \geqslant 1$. For the case $\alpha = 1$, Hardy* has shown that the function $H(s)$ is regular all over the plane, but the half-plane of summability by Riesz's typical means coincides with the half-plane of convergence. We proceed to show that the same is true, when $\alpha > 1$.

Our series is convergent if, and only if, $\sigma > 0$, and then absolutely convergent. It is not summable by Cesaro's means for any value of s whose real part is negative. For, if a series Σc_n is summable by Cesaro's means of order k, then

$$c_n = o(n^k);$$

and it is obvious that this condition is not satisfied in this case. Therefore†, the series is not summable (l, k) in the Riesz's sense for any value of s whose real part is negative. That is, the line of summability (l, k) is $\sigma = 0$, which is also the line of summability (λ, k).‡

It remains to prove that the series represents an integral function of s.

By Theorem VIII, we have

$$(18.1) \qquad H(s) = G(s) + \int_{1}^{\infty} e^{Ai(\log x)^{\beta} - s(\log x)^{\alpha}} dx,$$

$$= G(s) + \int_{0}^{\infty} e^{Aiy^{\beta} - sy^{\alpha} + y} dy = G(s) + J(s),$$

where $G(s)$ is an integral function of s, and the integral for $J(s)$ converges absolutely if $\sigma > 0$.

---

* 'Example to illustrate a point in the theory of Dirichlet's series', *loc. cit.*

† Hardy and Riesz : "The general theory of Dirichlet's series", *Cambridge Tracts in Mathematics*, 1915, Theorem 19.

‡ The last reference, Theorem 30.

Suppose for a moment that s is real and > 0. Let $A > 0$. Take y to be a complex variable and put $y = re^{i\theta}$. Choose a positive angle θ_1, such that $0 < \beta\,\theta_1 < \pi$, and $0 < \alpha\,\theta_1 < \frac{\pi}{2}$. Then the integral on the right-side of (18.1) may be taken along the radius vector $\theta = \theta_1$, provided only the integral

$$I \equiv \int_0^{\theta_1} e^{Ai\,(re^{i\theta})^{\beta} - s\,(re^{i\theta})^{\alpha} + re^{i\theta}}\, re^{i\theta}\, i d\theta \to 0, \text{ as } r \to \infty.$$

Now

$$|\,I\,| \leqslant \int_0^{\theta_1} r \exp.\ \{ -Ar^{\beta} \sin \beta\theta - sr^{\alpha} \cos \theta\alpha + r \cos \theta \}\, d\theta$$

$$\leqslant \theta_1\, re^{r - sr^{\alpha} \cos \alpha\,\theta_1},$$

which $\to 0$, as $r \to \infty$.

We, therefore, have, if s is real and > 0,

$$\text{(18.2)} \qquad J(s) = \int_0^{\infty} \exp.\ \{ Ai\,(re^{i\theta_1})^{\beta} - s(re^{i\theta_1})^{\alpha} + re^{i\theta_1} + i\theta_1 \}\, dr.$$

Now

$$\int_0^{\infty} \left| \exp.\ \{ Ai\,(re^{i\theta_1})^{\beta} - (\sigma + it)(re^{i\theta_1})^{\alpha} + re^{i\theta_1} + i\theta_1 \} \right| dr,$$

$$= \int_0^{\infty} \exp.\ \{ -Ar^{\beta} \sin \theta_1\beta + r^{\alpha}(-\sigma \cos \theta_1\alpha + t \sin \theta_1\alpha) + r\cos\theta_1 \}\, dr$$

which is uniformly convergent throughout any finite domain of the values of s. Therefore, the integral on the right-side of (18.2) represents an integral function of s.

Hence, from (18.1), H (s) is an integral function of s.

The conclusion remains true, when $A < 0$. In changing the path of integration of the integral on the right-side of (18.1), we must choose, in this case, $\theta_1 < 0$, and such that $|\,\theta_1\,\beta\,| < \pi$, and $|\,\theta_1\,\alpha\,| < \frac{\pi}{2}$.

I may add here that when $\beta > 1 + \alpha$, Theorem IX is immediately applicable.

Thus it has been shown that the series

$$(18.3) \qquad \sum_{1}^{\infty} e^{Ai(\log n)^{\beta} - s(\log n)^{\alpha}}, \qquad A \neq 0, \quad \beta > \alpha > 1;$$

$$(18.4) \qquad \sum_{1}^{\infty} e^{Ai(\log n)^{\beta}} n^{-s}, \qquad A \neq 0, \quad \beta > 1;$$

are similar* in two respects. Firstly, they both represent integral functions of s, and secondly, the half-plane of convergenee coincides, in each case, with the half-plane of summability by Riesz's typical means.

On the other hand the series

$$(18.5) \qquad \sum_{1}^{\infty} e^{Ain^{\beta} - s(\log n)^{\alpha}}, \quad \alpha > 1,$$

and

$$(18.6) \qquad \sum_{1}^{\infty} e^{Ain^{\beta}} n^{-s},$$

$$\left. \begin{array}{c} A \neq 0, \; 0 \angle \beta \angle 1, \\ \text{or} \\ \beta = 1, \text{ and } A \neq 2k\pi, \\ k = 0, 1, 2, 3, \ldots \ldots ; \end{array} \right.$$

though both represent integral functions of s, as was pointed out at the end of § 10, differ in this respect that while the series (18.5), like the series (18.3), has its half-plane of summability coincident with the half-plane of convergence, the series (18.6) is summable† by Riesz's typical means all over the plane.

---

* This raises another interesting question, 'how far the series (18.3) and (18.4) are similar as regards their summability (A).' It is known (Hardy: "The application of Abel's method of summation to Dirichlet's series" – *Quarterly Journal*, Vol. 47), that the series

$$\sum_{1}^{\infty} \exp. \{ Ai (\log n)^{\beta} \} n^{-s}, \; A > 0, \; \beta > 1,$$

is summable (A) all over the plane, if $\beta > 2$; summable (A) in the half-plane of convergence only, if $2 > \beta > 1$; and summable (A) if $\sigma > 1 - \pi A$, but not if $\sigma < 1 - \pi A$, when $\beta = 2$. It will be interesting to know whether any such results hold for the series (18.3).

So far I have not been able to throw any light on this point. Similarly, it will be interesting to consider the summability (A) of the series (18.5). My conjecture is that the series (18.3) and (18.5) are not summable (A) for any negative value of s.

† The same references as given at the end of § 10.

ON SOME IDENTITIES INVOLVING ZETA-FUNCTIONS.

BY S. D. CHOWLA.

Ramanujan has obtained the result*

$$\sum_{1}^{\infty} \frac{\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)},$$

where $\sigma_a(n)$ is the sum of the ath powers of the divisors of n. In this paper I consider certain series analogous to this, which can be evaluated in finite terms by means of Zeta-functions, and certain other kinds of functions.

§ 1. *The X-function and the L-function.*

We shall denote by $X_r(n)$ any arithmetical function of n which has the following properties:

(i) The value of $X_r(n)$ does not depend on the actual value of n, but only on its value modulo r; *i.e.*

$$X_r(n) = X_r(n+r).$$

(ii) $X_r(mn) = X_r(m)\,X_r(n)$ for any two positive integers m, n.

(iii) $X_r(n) = 0$ if n and r have a common factor, and is equal to ± 1 otherwise.

A case of the X-function is the extended Legendre-symbol $\left(\frac{n}{r}\right)$, whose value is 0, + 1, or − 1 according as n has a factor in common with r, or is a quadratic residue or non-residue of r, but for our purposes it is not necessary to particularise the X-function.

---

* *Messenger of Mathematics*, Vol. 45 (1916) pp. 81—84.

The L-function associated with an X-function is defined by

$$L_r(s) = \sum_{n=1}^{\infty} \frac{X_r(n)}{n^s}, \qquad \text{(real part of } s > 1).$$

From the properties of the X-function, it is clear that if p runs through all the primes for which $X_r(p) = 1$, and q runs through all the primes for which $X_r(q) = -1$, then

(1) $\{L_r(s)\}^{-1} = \Pi\,(1 - p^{-s})\,(1 + q^{-s}).$

I shall extend Ramanujan's result by introducing into the numerator $\sigma_a(n)\ \sigma_b(n)$ of his series, the functions $X_r(n)$, $\lambda(n)$, and also by replacing the σ-functions by certain other functions $\theta_a(n)$, $\theta_b(n)$, and evaluating the new series thus obtained.

The functions $\lambda(n)$, $\theta_a(n)$ are defined as follows:

$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p's are distinct primes,

$\lambda(n) = (-1)^{\alpha_1 + \alpha_2 + \dots \alpha_k}$

$\theta_a(n) = \Sigma\, \lambda(d)\, d^a$, summed over all divisors d of n.

§ **2.** If we confine ourselves to the operations of introducing the λ-function, and replacing the σ-function by the θ-function, we obtain five new series all of which can be easily evaluated in terms of Zeta-functions, by following Wilson's methods*. The actual values are:

(2) $$\sum_{1}^{\infty} \frac{\lambda(n)\,\sigma_a(n)\,\sigma_b(n)}{n^s}$$

$$= \frac{\zeta(2s)\,\zeta(2s-2a)\,\zeta(2s-2b)\,\zeta(2s-2a-2b)}{\zeta(2s-a-b)\,\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}.$$

* Proofs of some formulæ enunciated by Ramanujan, *Proc. Lond. Math Soc.* (1923) pp. 235 *et. seq.*

$$(3) \quad \sum_{1}^{\infty} \frac{\theta_a(n)\,\sigma_b(n)}{n^s}$$

$$= \frac{\zeta(s)\,\zeta(2s-2a)\,\zeta(s-b)\,\zeta(2s-2a-2b)\,\zeta(2s-a-b)}{\zeta(4s-2a-2b)\,\zeta(s-a)\,\zeta(s-a-b)}.$$

$$(4) \quad \sum_{1}^{\infty} \frac{\lambda(n)\,\theta_a(n)\,\sigma_b(n)}{n^s}$$

$$= \frac{\zeta(2s)\,\zeta(2s-2b)\,\zeta(2s-2a-2b)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(2s-a-b)}{\zeta(s)\,\zeta(s-b)\,\zeta(4s-2a-2b)\,\zeta(2s-2a-2b)}.$$

$$(5) \quad \sum_{1}^{\infty} \frac{\theta_a(n)\,\theta_b(n)}{n^s}$$

$$= \frac{\zeta(s)\,\zeta(2s-2a)\,\zeta(2s-2b)\,\zeta(s-a-b)}{\zeta(2s-a-b)\,\zeta(s-a)\,\zeta(s-b)}.$$

$$(6) \quad \sum_{1}^{\infty} \frac{\lambda(n)\,\theta_a(n)\,\theta_b(n)}{n^s}$$

$$= \frac{\zeta(2s)\,\zeta(2s-2a-2b)\,\zeta(s-a)\,\zeta(s-b)}{\zeta(s)\,\zeta(s-a-b)\,\zeta(2s-a-b)}.$$

These evaluations may be used to deduce several known results. For instance, it is easy to shew that

$$\sum \frac{\lambda(n)}{n} = \frac{\zeta(2s)}{\zeta(s)}; \quad \sum \frac{\theta_a(n)}{n^s} = \frac{\zeta(s)\,\zeta(2s-2a)}{\zeta(s-a)};$$

whence, putting $a = 0$, we obtain

$$\theta_0(n) = 1, \text{ if } n \text{ is a perfect square}$$
$$= 0, \text{ otherwise.}$$

Now, putting [illegible] in (3), we have

$$\ldots(n) = \sum \frac{\zeta^2(2s)}{\zeta(4s)}.$$

Hence,

(7) $$\sum \frac{\sigma_0(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}.$$

Also, Ramanujan has proved that:

(8) $$\sum \frac{[\sigma_0(n)]^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

(7) and (8) together give the known result

(9) $$\sum_{d/n} \sigma_0(d^2) = \{\sigma_0(n)\}^2.$$

Also, it is known that,

(10) $$\sum \frac{2^{\pi(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)},$$

where $\pi(n)$ is the number of different prime factors of n. Thus (7) and (10) give the known result:

(11) $$\sum_{d/n} 2^{\pi(d)} = \sigma_0(n^2).$$

Again, if we put $a = 0$ in (3), we obtain

$$\sum \frac{\Theta_0(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(2s)\,\zeta(2s-b)\,\zeta(2s-2b)}{\zeta(4s-2b)}.$$

Hence, substituting the value of $\Theta_0(n)$, we get

(12) $$\sum \frac{\sigma_b(n^2)}{n^s} = \frac{\zeta(s)\,\zeta(s-b)\,\zeta(s-2b)}{\zeta(2s-2b)}.$$

But, if we put $a = b$ in Ramanujan's formula, we obtain,

(13) $$\sum \left\{\frac{\sigma_b(n)}{n^s}\right\}^2 = \frac{\zeta(s)\,\{\zeta(s-b)\}^2\,\zeta(s-2b)}{\zeta(2s-2b)},$$

From (12) and (13) we have the known result:

(14) $$\sum_{d/n} d^b\,\sigma_b\left(\frac{n^2}{d^2}\right) = \{$$ [illegible]

This formula is the extension of (9).

Again, the formula

(15) $$\sum_{d/n} \{ \sigma_0(d) \}^3 = \{ \sum_{d/n} \sigma_0(d) \}^2,$$

re-discovered by Ramanujan, was originally obtained by Liouville as an immediate deduction from the Indian theorem :

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2.$$

Since we have seen that

$$\sum \frac{\sigma_0(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)},$$

it follows on multiplying both sides by $\zeta(2s)$, that

$$\sum_{d/n} \sigma_0(d) = \sum_{d^2/n} \sigma_0\left(\frac{n^2}{d^4}\right).$$

Taking account of (15), we have therefore the result :

(16) $$\sum_{d/n} \sigma_0{}^3(d) = \left\{ \sum_{d/n} \sigma_0(d) \right\}^2 = \left\{ \sum_{d^2/n} \sigma_0\left(\frac{n^2}{d^4}\right) \right\}^2.$$

§ 3. If, now, in Ramanujan's series and the five series of the last section, we introduce an X-function $X_r(n)$, we obtain six new series, all of which can be evaluated by following Wilson's method. The values of these series involve not only the associated L-funcrion L_r, but also certain arithmetical functions $Z_r(k)$ of the number r, defined by

$$Z_r(k) = \Pi\left(1 - \frac{1}{p^k}\right),$$

the product extending to all the prime divisors p of r.

The actual values are as follows :—

(17) $$\sum_{n=1}^{\infty} \frac{X(n)\,\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{L_r(s)\,L_r(s-a)\,L_r(s-b)\,L_r(s-a-b)}{Z_r(2s-a-b)\,\zeta(2s-a-b)}.$$

(18) $$\sum_{n=1}^{\infty} \frac{\lambda(n) X(n)\, \theta_a(n)\, \theta_b(n)}{n^s} = \frac{\zeta(2s)\, \zeta(2s-2a-2b)}{\zeta(2s-a-b)}$$

$$\times \frac{Z_r(2s)\, Z_r(2s-2a-2b)}{Z_r(2s-a-b)} \times \frac{L_r(s-a)\, L_r(s-b)}{L_r(s)\, L_r(s-a-b)}.$$

(19) $$\sum \frac{\lambda(n) X(n)\, \theta_a(n)\, \sigma_b(n)}{n^s} = \frac{\zeta(2s)\, \zeta(2s-2b)\, \zeta(2s-a-b)}{\zeta(4s-2a-2b)}$$

$$\times \frac{Z_r(2s)\, Z_r(2s-2b)\, Z_r(2s-a-b)}{Z_r(4s-2a-2b)} \times \frac{L_r(s-a)\, L_r(s-b-a)}{L_r(s)\, L_r(s-b)},$$

(20) $$\sum \frac{X(n)\, \theta_a(n)\, \theta_b(n)}{n^s} = \frac{\zeta(2s-2a)\, \zeta(2s-2a-2b)}{\zeta(2s-a-b)}$$

$$\times \frac{Z_r(2s-2a)\, Z_r(2s-2a-2b)}{Z_r(2s-a-b)} \times \frac{L_r(s)\, L_r(s-a-b)}{L_r(s-a)\, L_r(s-b)}.$$

(21) $$\sum \frac{X(n)\, \lambda(n)\, \sigma_a(n)\, \sigma_b(n)}{n^s}$$

$$= \frac{\zeta(2s)\, \zeta(2s-2a)\, \zeta(2s-2b)\, \zeta(2s-2a-2b)}{\zeta(2s-a-b)}$$

$$\times \frac{Z_r(2s)\, Z_r(2s-2a)\, Z_r(2s-2b)\, Z_r(2s-2a-2b)}{Z_r(2s-a-b)}$$

$$\times \frac{1}{L_r(s)\, L_r(s-a)\, L_r(s-b)\, L_r(s-a-b)}.$$

(22) $$\sum \frac{X(n)\, \theta_a(n)\, \theta_b(n)}{n}$$

$$= \frac{\zeta(2s-2a)\, \zeta(2s-2a-2b)\, \zeta(2s-a-b)}{\zeta(4s-2a-2b)}$$

$$\times \frac{Z_r(2s-2a)\, Z_r(2s-2a-2b)\, Z_r(2s-a-b)}{Z_r(4s-2a-2b)}$$

$$\times \frac{L_r(s)\, L_r(s-b)}{L_r(s-a)\, L_r(s-a-b)}.$$

(23) $$\sum \frac{\lambda(n)\, X_r(n)\, \theta_a(n)}{n^s} = \zeta(2s)\, Z_r(2s) \,.\, \frac{L_r(s-a)}{L_r(s)}$$

We can establish these series by expressing the zeta-functions, the Z-functions as well as the L-functions as products (1). In the first place it will be obvious that when we express the right-side in any of (17) to (23) as a zeta-series, there will occur no term in $\frac{1}{N^s}$, if N has a common factor with r. Let now p be any prime which does not occur in r; write $t = \pm p^{-s}$ according as $X_r(p) = \pm 1$, also write x for p^a and y for p^b.

Now, considering (17), for instance, the part of the infinite product expression of the right side involving the prime p is:

$$\frac{(1-xyt^2)}{(1-t)(1-xt)(1-yt)(1-xyt)} = (1-xyt^2)\,(\Sigma\, P_k t^k),$$

where P_k is the sum of the homogenous products of k dimensions, of the four quantities, $1, x, y, xy$. To obtain P_k, we have the decomposition

$$\frac{1}{(1-xt)(1-yt)(1-xyt)} = \frac{A}{1-xt} + \frac{B}{1-yt} + \frac{C}{1-xyt},$$

where

$$A = \frac{x}{(x-y)(1-y)},\quad B = \frac{y}{(y-x)(1-x)},\quad C = \frac{xy}{(1-x)(1-y)}.$$

Therefore, the homogenous product-sum, Q_k of k dimensions of the three quantities x, y, xy is:

$$Q_k = Ax^k + By^k + Cx^ky^k$$

$$= (h_k - h_{k+1} + x^{k+1}y^{k+1})/(1-x)(1-y),$$

where $$h_k = x^k + x^{k-1} + \dots + y^k.$$

Hence $$P_k = Q_0 + Q_1 \dots + Q_k$$

$$= \frac{1 - h_{k+1} + xy + x^2y^2 \dots + x^{k+1}y^{k+1}}{(1-x)(1-y)} = R_{k+1} + xy\,P_{k-2}.$$

where $$R_{k+1} = \frac{(1-x^{k+1})(1-y^{k+1})}{(1-x)(1-y)}.$$

Thus $P_k - xy\, P_{k-2}$, and therefore the co-efficient of t^k in the part of the infinite product on the right is R_{k+1}. Remembering that $t = \pm p^{-s}$ according as $X_r(p) = \pm 1$, it follows at once that the co-efficient of $\frac{1}{N^s}$ on the right of (17) is,

$$\Pi \frac{(p^{a(\alpha+1)} - 1)(p^{b(\alpha+1)} - 1)}{(p^a - 1)(p^b - 1)} \times \Pi (-1)^\beta \frac{(q^{a(\beta+1)} - 1)(q^{b(\beta+1)} - 1)}{(q^a - 1)(q^b - 1)}.$$

where $$N = p^\alpha\, p'^{\alpha'} \ldots q^\beta\, q'^{\beta'} \ldots,$$

p denoting the prime factors for which $X_r(p) = 1$, and q denoting those for which $X_r(q) = -1$. That is, the co-efficient of $\frac{1}{N^s}$ on the right is $X_r(N)\, \sigma_a(N)\, \sigma_b(N)$, proving (17).

We may prove the remaining equalities in the same way.

§ 4. We may use the same method to prove the following general theorem, which is very useful.

If $f_r(n)$ denote the excess of the number of those divisors d of n, for which $X_r(d) = +1$, over the number of the divisors d' for which $X_r(d') = -1$, then

$$(24) \qquad \sum \frac{f_r(n)}{n^s} = \zeta(s)\, L_r(s).$$

This may also be seen directly, since the co-efficieut of $\frac{1}{n^s}$ in the product $\zeta(s)\, L_r(s)$ must be $\sum_{d/n} X_r(d)$, that is, $f_r(n)$.

As an example of a series involving $f_r(n)$, we can show that:

$$(25) \qquad \sum \frac{f^2_r(n)}{n^s} = \frac{\zeta^2(s)\, L^2_r(s)\, Z_r(s)}{\zeta(2s)\, Z_r(2s)}.$$

To prove this, write $t = p^{-s}$. If the prime p is a factor of r, the part of the infinite product on the right, which involves p, reduces to $\frac{1}{1-t}$. If $X_r(p) = +1$, the part of the product involving p is

$$\frac{1-t^2}{(1-t)^2(1-t)^2} = \Sigma (k+1)^2\, t^k.$$

Lastly, if $x_r(p)$ is -1, the part of the product will be:

$$\frac{1-t^2}{(1-t)^2(1+t)^2} = \frac{1}{1-t^2} = \Sigma\, t^{2k}.$$

These together, evidently establish (25).

§ 5. *The function R (n).*

According to a classical result of Gauss, any number $n = 2^{\mu}MN$, where the prime factors in M, N are of the respective forms $4k-1$, $4k+1$, can be expressed as a sum of two squares only when M is a perfect square, and that in $\frac{1}{2}d(N)$ or $\frac{1}{2}\{d(N)+1\}$ ways, according as N is not, or is a perfect square ($d(N)$ representing the number of divisors of N). In this enumeration, the representations a^2+b^2, b^2+a^2 of n are not reckoned as different. If however, we agree to reckon them as different, the number of representations becomes $d(N)$ and $d(N)+1$ in the two cases. If, further, for the case in which N is a square, we agree that either of the following possible representations is to be reckoned only once, namely,

$$n = (2^m\sqrt{M}\sqrt{N})^2 + (2^m\sqrt{M}\sqrt{N})^2 \quad (\mu = 2m+1),$$

$$n = (2^m\sqrt{M}\sqrt{N})^2 + 0^2 = 0^2 + (2^m\sqrt{M}\sqrt{N})^2 \quad (\mu = 2m),$$

then we can say that in all cases the number of representations of n as a sum of two squares is given by

$$R(n) = d(N).$$

This is equivalent to the alternative definition of $R(n)$ as the co-efficient of x^n in $(\frac{1}{2} + x + x^4 + x^9 + \dots)^2$.

The fact that $R(n) = d(N)$ when M is a perfect square, and $= 0$ otherwise, shows that $R(mn) = R(m)\,R(n)$, whenever m and n have no common prime factor of the form $4k+1$, in particular when m and n are mutually prime.

An important property of $R(n)$ is that *$R(n)$ is also equal to the excess of the number of divisors of n of the form $4k+1$, over those of the form $4k-1$.*

To prove this we have to observe that this excess is equal to

$$d\,(\mathrm{N}) \times \sum_{d/\mathrm{M}} \lambda\,(d).$$

But the latter factor is known to be equal to 0 or 1 according as M is not, or is a perfect square. Hence the excess is equal to d (N) or 0, according as M is not, or is a square; that is, it is equal to R (n).

If we define an x-function, by

$$x\,(2) = 0;\; x\,(p) = \pm 1,$$

according as the prime p is of the form $4k+1$, or $4k-1$, then the corresponding L-function is

$$\eta\,(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \ldots\ldots$$

Also from what has been proved, R (n) is the excess of the number of divisors d of n, for which $x\,(d) = +1$, over those for which $x\,(d) = -1$. Hence by (24),

$$(26) \quad \sum \frac{\mathrm{R}\,(n)}{n^s} = \zeta\,(s)\,\eta\,(s)$$

$$(27) \quad \sum \frac{\mathrm{R}^2\,(n)}{n^s} = \frac{\zeta^2\,(s)\,\eta^2\,(s)\,(1-2^{-s})}{\zeta\,(2s)\,(1-4^{-s})} = \frac{\zeta^2\,(s)\,\eta^2\,(s)}{\zeta\,(2s)\,(1+2^{-s})}$$

We can prove further that:

$$(\text{-}8) \quad \sum \frac{\mathrm{R}\,(n^2)}{n^s} = \frac{\zeta^2\,(s)\,\eta\,(s)}{(1+2^{-s})\,\zeta\,(2s)}.$$

To prove this, write $t = p^{-s}$; then the part involving p in the infinite product on the right is

$$\frac{1+t}{(1-t)^2} = \Sigma\,(2k+1)\,t^k = \Sigma\,\mathrm{R}\,(p^{2k})\,t^k, \text{ if } p = 1 \bmod 4$$

$$\frac{(1-t^2)}{(1-t)^2\,(1+t)} = \Sigma\,t^k = \Sigma\,\mathrm{R}\,(p^{2k})\,t^k, \text{ if } p = -1 \bmod 4$$

$$\frac{(1-t^2)}{(1-t)^2\,(1+t)} = \Sigma\,t^k = \Sigma\,\mathrm{R}\,(2^{2k})\,t^k, \text{ if } p = 2.$$

It will be noticed that (27) and (28) imply:

$$(29) \quad \sum_{\delta/n} R(\delta^2)\, a\left(\frac{n}{\delta}\right) = R^2(n)$$

We may also prove that

$$(30) \quad \sum \frac{\lambda(n)\, R^2(n)}{n^s} = \frac{\zeta^3(2s)\,(1+2^{-s})\,(1-2^{-2s})}{\zeta^2(s)\,\eta^2(s)}$$

From (27) and (30) we have:

$$\sum_{d/n} \lambda(d)\, R^2(d)\, R^2\left(\frac{n}{d}\right) = \text{co-efficient of } \frac{1}{n^s} \text{ in } \zeta^2(2s)\,(1-2^{-2s})$$

= 0, if n is not a square and equal to $d(N)$, if n is a square, and N^2 is its largest odd factor.

ON SOME EMPIRICAL THEOREMS OF SCHERK.*

By S. Sivasankaranarayana Pillai.

§ 1. Let

$$p_1, p_2, \dots, p_n,$$

be the first natural primes in order so that $p_1 = 2$, $p_2 = 3$, etc. Then Scherk's theorems are:

Theorem I. If n is even, p_n can be obtained by the addition and subtraction of all the smaller primes and unity, each taken once,

e.g., $\quad p_8 = 19 = 1 + 2 + 3 + 5 - 7 + 11 - 13 + 17.$

Theorem II. If n is odd, p_n can be obtained by addition and subtraction of all the primes till p_{n-2} (inclusive) and unity, each taken once, together with the addition of $2p_{n-1}$,

e.g., $\quad p_7 = 17 = -1 + 2 + 3 + 5 - 7 - 11 + 2.13.$

These two conjectures by Scherk, were published in a paper in the *Journal fur Math.*, Vol. 10, (1883) (pp. 201—208) in which he explained how he had arrived at them, and why he had to give different results for n odd and n even. After discussing one mode of representation, assuming that such a representation possible, the author derives

Theorem III.

(i) $1 - p_1 + p_2 - \dots + p_{2n-2} - p_{2n-1} + 2p_{2n} > p_{2n+1}$.

(ii) $1 - p_1 + p_2 - \dots - p_{2n-1} + p_{2n} + p_{2n+1} > p_{2n+2}$

In general problems involving the expression of a number as the sum of primes are exceedingly difficult. There is the classical theorem of Goldbach that every even integer is the sum of two primes and Kronecker's result that every even integer can be expressed as the difference of two primes in an infinity of ways. The progress towards

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* This problem was suggested to me by Dr. Vaidyanathaswami.

the solution of these problems has been exceedingly slow and has taxed the utmost resources of the analytical theory of numbers.

The theorems discussed here belong, however, to a different class and involve the expression of a number, not as the sum or difference of a *fixed number* of primes, but of *all* the lower primes. The results are in fact not at all deep nor difficult to establish. Theorems I and II are proved in a very elementary manner. In the proof of Theorem III, I have used the first three terms in the asymptotic expansion of $\pi(x)$ the number of primes less than x.

§ 2. LEMMA (1) BERTRAND'S POSTULATE:

Between x and $2x - 1$, when $x \geqslant 2$ there is at least one prime.

This was first proved by Tchebicheff, the proof being entirely elementary and very elegant. Landau and Ramanujan have also given proofs; but the essential principle of their proofs is only Tchebicheff's, Ramanujan's being the simplest of all.

It may be interesting to note that this theorem follows immediately from the unproved Goldbach's theorem.

LEMMA (2) *If n is any positive integer, n can be represented as the sum of different positive primes (1 also being regarded as a prime).*

For let $p(n)$ denote the greatest prime not exceeding n and let $n - p(n) = n_1$, $n_1 - p(n_1) = n_2$ and so on, so that by addition

$$n = p(n) + p(n_1) + \dots + p(n_r) + n_{r+1} \qquad \dots (1)$$

The sequence (n_r) is a decreasing sequence of positive integers; hence for some value of r we must have $n_{r+1} = 0$.

$$\therefore \qquad n = p(n) + p(n_1) + \dots + p(n_r).$$

Also by (1), the primes $p(n)$ are all different. Hence the lemma.*

It may be noted that the above representation is unique. If $R(x)$ is the number of primes required to express x in the form (1), then it can

* This lemma and the proof are due to Prof. K. Ananda Rao.

be proved that:

(i) there are numbers for which $R(x)$ can be as great as we please;

(ii) $\frac{R(x)}{\log x} \to 0$;

(iii) $R(x) < 2 \log \log x$, on Riemann's hypothesis.

It is very likely $R(x)$ is less than any logarithmic function of x. The first four numbers for which $R(x)$ is maximum are

$$1 = 1$$
$$4 = 3 + 1$$
$$27 = 23 + 3 + 1$$
$$1354 = 1327 + 23 + 3 + 1;$$

The 5th number probably exceeds 100000.

LEMMA (3).

If $S_n = 1 + p_1 + p_2 + \dots + p_n$,

then $S_n \geqslant p_{n+1}$ when $n \geqslant 2$.

The proof follows directly by induction and the use of Bertrand's postulate.

§ 3. *Proof of Theorem I.*

It the representation be possible, let the sum of the primes with positive signs be x and the sum of the rest be y, unity being treated as a prime.

Then if $S_n = 1 + p_1 + p_2 + \dots + p_n$:

$$S_{n-1} = x + y; \quad p_n = x - y.$$

$$\therefore \quad 2x = S_{n-1} + p_n = S_n.$$

$$\therefore \quad x = \frac{S_n}{2}.$$

Since n is even, S_n is even and $\frac{S_n}{2}$ is an integer. Such a representation is not possible when n is odd, since $\frac{S_n}{2}$ is not an integer.

Choose r so that

$$S_{n-1} - S_{n-1-r} \leqslant x < S_{n-1} - S_{n-r-2} \qquad \dots (2)$$

and let $$x - (S_{n-1} - S_{n-1-r}) = a.$$

Then $$x = p_{n-1} + p_{n-2} + \dots p_{n-r} + a.$$

Now by the inequality (2)

$$a < p_{n-r-1}$$

and by lemma (2), a can be represented as the sum of different primes. Each of these will be less than p_{n-r-1}. Therefore x can be represented as the sum of distinct primes each less than p_n. But by definition

$$y = S_{n-1} - x.$$

Hence in y, all the rest of the primes come in and hence the theorem.

Proof of Theorem II:

Assuming the representation is possible, let the sum of the terms with positive signs be equal to x, and sum of the rest be y.

Then $$p_n = x - y; \text{ and } p_{n-1} + S_{n-1} = x + y.$$

$$\therefore \quad x = \frac{S_n + p_{n-1}}{2}.$$

since n is odd, $\frac{S_n + p_{n-1}}{2}$ is an integer, which is necessary.

By lemma (3), $x - 2p_{n-1}$ is positive. The rest of the proof follows as in I.

Proof Theorem III.

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{\log^3 x}\right)$$

$$\therefore \quad \pi\left(x\left(1 + \frac{1}{\log x}\right)\right) = \frac{x}{\log x} + \frac{2x}{(\log x)^2} + O\left(\frac{x}{(\log^3 x)}\right)$$

$$\therefore \quad \pi\left(x\left(1 + \frac{1}{\log x}\right)\right) - \pi(x) = \frac{x}{(\log x)^2} + O\left(\frac{x}{\log^3 x}\right).$$

Therefore, when x is large enough, between x and $x\,(1 + 1/\log x)$, there are as many primes as we wish. ... ... (4)

Put
$$p_{2n} = x.$$
Then
$$p_{2n} + p_{2n+1} > 2x.$$
Now
$$|\,1 - p_1 + p_2 - p_3 + \ldots + p_{2n-2} - p_{2n-1}\,|$$
$$< x + p_1 - p_2 + p_3 - p_4 + \ldots + p_{2n-1} - p_{2n}$$

(by adding and subtracting p_{2n}).

$$< x - 2 \,.\, \frac{x}{2\log x} \text{ i.e. } < x - \frac{x}{\log x}.$$

since, $p_4 - p_3 \geqslant 2$, $p_6 - p_5 \geqslant 2$, etc., and there are at least $\frac{x}{2\log x}$ such intervals $\left(\text{since } \pi(x) > \frac{x}{\log x}\right)$.

$\therefore$
$$1 - p_1 + p_2 - \ldots + p_{2n-1} - p_{2n-1} + 2p_{2n}$$
$$> 2p_{2n} - |\,1 - p_1 + p_2 + \ldots - p_{2n-1}\,|$$
$$> 2x - \left(x - \frac{x}{\log a}\right)$$
$$= x\left(1 + \frac{1}{\log x}\right)$$
$$> p_{2n+2} \text{ by (4).}$$

Hence the theorem.

§ 4. Now we can generalise theorems I and II into

THEOREM IV. If a is p_{2n}, or any even integer between p_{2n} and p_{2n+1}, or any odd integer between p_{2n-1} and p_{2n} (p_{2n-1} excluded), then a can be obtained by the addition and subtraction of all the primes $< a$ together with unity, each taken once.

e.g. $p_4 < 8 < p_5 \quad \therefore 8 = 1 - 2 - 3 + 5 + 7$

$$p_9 < 25 < p_{10}$$

$$\therefore \; 25 = 1 + 2 - 3 + 5 - 7 - 11 + 13 - 17 + 19 + 23.$$

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

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Notes and Questions.

Probability and Pure Mathematics.

'Concrete Probability' in its obviously interesting senses is not a mathematical idea at all, but an idea of physics or philosophy. The 'mathematical theory of probability' is merely a collection of analytical theorems which (so far as mathematicians can tell) may or may not have any application.

The mathematical thoory consists essentially of two parts :—

(1) the entirely finite theory, which is merely permutations and combinations stated in a different language;

(2) the higher theory, which in addition to (1), includes problemms about limits. Even in (1) it is common to state theorems as if the 'concrete idea' of probability was relevant, *e.g.*, we talk about certain elementary cases as being 'considered as equally likely'—but this is meaningless. At bottom, mathematically, the theorem

'*the chance of drawing an ace is* $\frac{1}{13}$.'

means simply that there are 52 cards and 4 aces and nothing else whatever.

When we come to statements involving probability and infinite aggregates, *everything* is a matter of definition. For example, if we wish to define the meaning of such a proposition as

'*the chance that a number is odd is* $\frac{1}{2}$'

it is natural to define that

"'*the probability that n has the property P is the limb of* e_N/N *when* $N = \infty$, *where* e_N *is the number of numbers less than N with the property P.*"

—but of course this is only one definition among many.* We might

* The definition given simply states a property of a particular sequence (*viz.*, the sequence of integers in increasing order of magnitude) which enumrates

also *define* the probability that a number is divisible by p as the reciprocal of the number of residues mod. p. This would give the same result [a genuine but trivial theorem]. Similarly the probability that a rational number is in its lowest terms may be defined as being

$$\lim_{N \to \infty} \frac{p_N}{N}$$

where it is presupposed that the rationals are enumerated in a definite order, (*e.g.*, by the Farey Series) and p_N denotes the number of rationals in their lowest terms from among the first N rationals.

Here, naturally there is no arrangement which is obviously pre-eminent, and so no single definition which stands out as eminently natural. It follows from these general considerations that in any mathematical theorem in which the word probability occurs the word can be removed by substituting the defining phrase, without any change in the meaning of the theorem (or of the proof).

And in fact this can always be done without inconvenience except in 'the mathematical theory of probability' itself, where it would lead to intolerable tediousness.

For example, in the theory of measure, we may define the probability that a number in (0,1) has a certain property as the measure of the appropriate set. So a lot of the theorems of measure can be stated in terms of probability, [*e.g.*, Diophantine Approximation], but, as a rule nothing particular is gained by doing so.

Proofs have often been offered of theorems in the analytic theory of numbers, which appear to depend on probability. These are of two classes. The argument is always that,

(A) We show in one way the probability of something is x,

(B) in another way that it is y

and deduce that $x = y$. These are two possibilities.

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all numbers. If we consider the sequence

1, 3, 5, 2, 7, 9, 11, 4, 13 15, 17, 6 19, 21, 23, 8, ...

as one which enumerates all positive integers then we can say that '*the probability that a number chosen at random is even is* $\frac{1}{4}$.

Either (α) the definitions of probability presupposed in (A) and (B) are the same. Or (β) they are different. In case (α) the proof is valid, but nothing has been gained by the introduction of the idea. In case (β) the proof is invalid. The real theorem is that the probability by first definition is equal to the probability by the second definition and this is precisely the same theorem which we set out to prove.

For example, let us consider the note published sometime back in this Journal* where the ideas of probability are brought in, in an attempt to prove that

$$\lim_{n \to \infty} \frac{1}{n} \left(\sum_{r=1}^{n} \frac{\phi(r)}{r} \right) = \frac{6}{\pi^2}.$$

In the beginning of that proof (p. 164, line 17) it is stated that 'hence the chance that a is prime to b in this case is $\phi(r)/r$.' This statement is either, merely a definition, or it refers to a particular property of a sequence of fractions the numerator of each of which is equal to r. In either case it is not at all clear how we can deduce what is said in lines 18—21 of the same page from what precedes the statement. For, all that has been said before simply amounts to either, a series of definitions or to a statement of certain properties of n separate sequences such as, say,

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \quad \frac{1}{m}, \ldots \ldots$$

$$\frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots \quad \frac{2}{m}, \ldots \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{n}{1}, \frac{n}{2}, \frac{n}{3}, \ldots \quad \frac{n}{m} \ldots \ldots$$

Even if we agree to associate some meaning to that sentence by making it refer to a property of some sequence of fractions (made up of the preceding sequences), say the sequence

$$\frac{1}{1}, \frac{2}{1}, \ldots \frac{n}{1}, \frac{2}{1}, \frac{2}{2}, \ldots \frac{n}{2}, \frac{1}{3}, \frac{2}{3}, \ldots \frac{n}{3}, \ldots$$

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* Vol. 16. pp. 164—165.

—even if we agree to do that—it only makes it altogether impossible to interpret the conclusion that follows concerning a sequence made up of *all* rational numbers. I do not see how, in a natural way, we can evolve a sequence consisting of all rational numbers such that, for that sequence, the chance for a/b being in the lowest terms is equal to

$$\lim_{n=\infty} \frac{1}{n}\left(\frac{\phi(1)}{1}+\frac{\phi(2)}{2}+\ldots+\frac{\phi(n)}{n}\right),$$

—supposing for a moment that this limit exists.

Similar remarks apply to what is said on p. 165. If what is, stated there is not to be regarded as only a series of arbitrary definitions, then every line in that page should be connected with a previously given sequence of the rationals. Further, *each* result of the multiplication of probabilities, and the limiting case of the infinite product among probabilities should be justified by a discussion based on the nature of the particular sequence considered. Finally the identity between this sequence and the sequence which we have to assume in order to interpret the conclusion on p. 164, should be established. It is only after doing all this that the proof given will be valid.

T. VIJAYARAGHAVAN.

Remarks by Mr. A. Narasinga Rao.

My object in making these remarks is, to point out that the word probability is used in a definite sense in my note "On the average value of $\phi(n)/n$" and that the argument developed there admits of being readily thrown into a form in which it is purely a question of the evalmation of a limit in two different ways.

Let us denote by $\{m, n\}$ the number of pairs of numbers, which are relatively prime, of which one is less than m and the other less than n The probability that two numbers taken at random are relatively prime may be defined as the limit of $\{m, n\}/mn$ when m and n tend to infinity. It remains to specify the manner in which m and n become infinite. In my note I have supposed that one of the variables, say m tends to infinity first, and then n.

We may arrange the mn pairs of numbers in a rectangular matrix formation with n rows of which the typical row is:—

$$(r, 1)\ (r, 2)\ \ldots\ \ldots\ \ldots\ (r, m) \quad (r = 1, 2, \ldots n)$$

Let us denote the number of relatively prime pairs in this row by m_r so that

$$\{m, n\} = m_1 + m_2 + \dots m_n.$$

$$\therefore \qquad \frac{\{m, n\}}{mn} = \sum_r \frac{1}{n} \cdot \frac{m_r}{m}$$

Now when m tends to infinity the limir m_r/m is $\phi(r)/r$, since in any complete residue class modulo r there are $\phi(r)$ members prime to r. Hence

$$\psi(n) = \underset{m \to \infty}{\text{Lt}} \frac{\{m, n\}}{mn} = \frac{1}{n}\left(\frac{\phi(1)}{1} + \frac{\phi(2)}{2} + \dots \frac{\phi(n)}{n}\right) \quad \dots \quad (1)$$

It is known* that $\psi(n)$ tends to a definite limit when $n \to \infty$; let us denote this by β so that

$$\beta = \underset{n \to \infty}{\text{Lt}} \psi(n) \quad \dots \quad \dots \quad (2)$$

and β is the average value of $\frac{\phi(n)}{n}$.

But we may also group the mn pairs of numbers in a different manner by bringing together those pairs which have the same highest common factor f. If (a, b), $a < m$, $b < n$ be such a pair then $a = a'f$, $b = b'f$ where a' and b' are integers which are relatively prime. Hence the number of such pairs (a, b) whose highest common factor is f is

$$\left\{\left[\frac{m}{f}\right], \left[\frac{n}{f}\right]\right\}$$

where as usual $[x]$ stands for the greatest integer less than x. Since f may take all values from 1 to n, we have

$$mn = \{m, n\} + \sum_{f=2}^{n} \left\{\left[\frac{m}{f}\right], \left[\frac{n}{f}\right]\right\}$$

$$\therefore \qquad 1 = \frac{\{m, n\}}{mn} + \frac{1}{mn} \sum_{f=2}^{n} \left\{\left[\frac{m}{f}\right], \left[\frac{n}{f}\right]\right\} \quad \dots \quad (3)$$

When m becomes infinite the limit of the first term is $\psi(n)$ by definition

* *Encyclopédie des Sciences Mathematiques Tome. I*, Vol. 3, p. 362.

and that of the general term is

$$\underset{[m/f]\to\infty}{\mathrm{Lt}} \frac{\{[m/f],\,[n/f]\}}{[m/f]\,.\,[n/f]} \;.\; \underset{m\to\infty}{\mathrm{Lt}} \frac{[m/f]}{m} \,.\, \frac{[n/f]}{n}$$

$$= \psi\,[n/f]\,.\,\frac{1}{f}\,.\,\frac{[n/f]}{n} \text{ by (1).}$$

Making n now infinite we have from (2) and (3)

$$1 = \beta + \beta \sum_{f=2}^{\infty} \frac{1}{f^2} = \beta\,.\,\frac{\pi^2}{6}$$

whence $$\frac{6}{\pi^2} = \beta = \text{the average value of } \frac{\phi(n)}{n}$$

It is not quite clear whether we shall obtain the same limit for $\{m, n\} / mn$ if m and n tend to infinity in a different manner. Since $\{m, n\}$ is a symmetric function of m and n the result will be the same if we make $n \to \infty$ first and later m. Prof. Weitzenbock has calculated* the limit in question with $m = n$ both tending to infinity, and obtains the same value $6/\pi^2$. This makes it likely that the limit is independent of the manner in which m and n tend to infinity. It would be interesting to get more definite information on this point.

(2,1) Correspondence between two Complex Planes.

1. Two variables z and w are in (2,1) correspondence when they satisfy an equation of the form

$$z^2(a'w - a) + z(b'w - b) + (c'w - c) = 0 \quad \ldots \quad (1)$$

or $$w = \frac{az^2 + bz + c}{a'z^2 + b'z + c'} \equiv \mathrm{R}(z) \text{ say.} \ \ldots \quad \ldots \quad (2)$$

In this note, it is shown that every (2,1) correspondence can be regarded as made up of homographic transformations and the transformation in any particular (2, 1) correspondence. The geometrical relation between the two "companion points" z_1 and z_2 which correspond to the same w, is obtained, as also the forms of some corresponding curves in the two planes.

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* "Ueber die Wahrscheinlichkeit dass zwei naturlistic Zahlen relative prime sind." *Koninklijke Akademie van Wetenschappen te Amsterdam.* This paper was not available for reference, but a reprint was kindly secured for me by Dr. N. G. H. W. Beeger of Amsterdam.

2. *Canonical Form.*

It will first be proved that the transformation in any (2,1) correspondence is the product of homographic transformations and the transformation $w = z^2$.

To prove this, we notice that there are two value of λ, say λ_1 and λ_2, such that

$$(az^2 + bz + c) + \lambda\,(a'z^2 + b'z + c')$$

is a perfect square. Hence from (2)

$$\frac{w + \lambda_1}{w + \lambda_2} = \frac{(az^2 + bz + c) + \lambda_1 (a'z^2 + b'z + c')}{(az^2 + bz + c) + \lambda_2 (a'z^2 + b'z + c')} = k\,\frac{(z - \alpha_1)^2}{(z - \alpha_2)^2} \quad \ldots \quad (3)$$

where k, α_1 and α_2 are determinate constants.

On writing

$$H_1(z) = \frac{z - \alpha_1}{z - \alpha_2};\quad H_2^{-1}(w) = \frac{1}{k}\,\frac{w + \lambda_1}{w + \lambda_2};\ \text{and } Q(z) = z^2,$$

we have

$$H_2^{-1}(w) = QH_1(z)$$

or

$$R(z) = w = H_2\,QH_1(z) \quad \ldots \quad \ldots \quad (4)$$

where H_1 and H_2 are homographic transformations.

It will now be shown that in the above result, we may take any particular (2, 1) transformation in place of $w = z^2$.

Let S be any such transformation. We have seen that S may be expressed in the form

$$S = H_4\,Q\,H_3 \quad \ldots \quad \ldots \quad (5)$$

where H_3 and H_4 are homographies.

$$\therefore \quad Q = H_4^{-1}\,S\,H_3^{-1}.$$

Substituting this in (4) we have

$$R = H_2\,H_4^{-1}\,S\,H_3^{-1}\,H_1 = H_5\,S\,H_6$$

since the product of two homographic transformations is another homographic transformation.

This is of the same form as (4) with S substituted for Q.

In particular we may take

$$S(z) = w = \tfrac{1}{2}\left(z \pm \frac{1}{z}\right).$$

3. *Companion Points.*

Companion points, *i.e.*, the two points z corresponding to the same w are obviously in involution. If z_1 and z_2 be such a pair of points,

$$w = \frac{az_1^2 + bz_1 + c}{a'z_1^2 + b'z_1 + c'} = \frac{az_2^2 + bz_2 + c}{a'z_2^2 + b'z_2 + c'}$$

$$\therefore \quad (ab' - a'b) z_1 z_2 + (ac' - a'c)(z_1 + z_2) + bc' - b'c = 0$$

$$\therefore \quad \left(z_1 + \frac{ac' - ca'}{ab' - ba'}\right)\left(z_2 + \frac{ac' - ca'}{ab' - ba'}\right) = \text{constant.}$$

So if Z_1, Z_2 and K are the points whose affixes are z_1, z_2 and $\frac{ca' - ac'}{ab' - ba'}$.

then $\overline{KZ_1} \times \overline{KZ_2}$ is constant.

$\therefore$ $KZ_1 \times KZ_2$ is constant, and the bisector of the angle Z_1KZ_2 is contant in direction.

4. *Corresponding Curves in the two Planes.*

From the relation (3) it is seen that if $\frac{z - \alpha_1}{z - \alpha_2}$ has its modulus or argument constant, then $\frac{w + \lambda_1}{w + \lambda_2}$ has also its modulus or argument constant. So the two systems of orthogonal co-axal circles in the z-plane with α_1, α_2 as limiting and common points respectively correspond in the w-plane to the two systems of co-axal circles with $-\lambda_1$ and $-\lambda_2$ as limiting and common points.*

Taking any circle in the z-plane, we shall now find the corresponding curve in the w-plane. First consider the transformation $w = z^2$.

Let $z = x + iy$; $w = \rho(\cos\theta + i\sin\theta) = u + iv$

$$\therefore \quad z = w^{\frac{1}{2}} = \pm \rho^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right).$$

Now write $x = \pm \rho^{\frac{1}{2}}\cos\frac{\theta}{2}$ $\quad y = \pm\rho^{\frac{1}{2}}\sin\frac{\theta}{2}$

in the equation of any circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

* I am indebted to Mr. A. Narasinga Rao for drawing my attention to this result.

and we get

$$\rho \pm 2\rho^{\frac{1}{2}} \left(g \cos\frac{\theta}{2} + f \sin\frac{\theta}{2} \right) + c = 0.$$

$$\therefore \quad \left(\rho + c \right)^2 = 4\rho \left(g \cos\frac{\theta}{2} + f \sin\frac{\theta}{2} \right)^2.$$

$$\therefore \quad \{ \rho^2 + c^2 - 2 (g^2 - f^2) \cos\theta - 2fg\, l \sin\theta \}^2$$
$$= 4\rho^2 (g^2 + f^2 - c)^2. \quad \ldots \quad (6)$$

Since $\rho \cos\theta = u$ and $\rho \sin\theta = v$,

$$\{ u^2 + v^2 + c^2 - 2 (g^2 - f^2) u - 2fgv \}^2$$
$$= 4 (u^2 + v^2) (g^2 + f^2 - c)$$

which is the equation of a bi-circular quartic.

We shall now extend the above result to any (2,1) transformation.

By (4), $R = H_2QH_1$.

Taking any circle in the z-plane, H_1 transforms it into another circle. Then Q transforms the circle into a bi-circular quartic as seen above.

Now it is a classical result that every homographic transformation is the product of two inversions; also the inverse of any bicircular quartic is another bicircular quartic. It follows therefore, that any (2, 1) transformation carries any circle in the z-plane into a bicircular quartic in the w-plane.

5. Corresponding to $z = \alpha_1$ and $z = \alpha_2$ we have $w = -\lambda_1$ and $w = -\lambda_2$. These are branch points on the two-sheeted Riemann surface representing w and are the extremities of the line of junction of the two sheets. If two curves cut at an angle θ at α_1 or α_2 the corresponding curves meet at an angle 2θ at $-\lambda_1$ or λ_2 on the w-plane. Except at these points the correspondence is conformal.

S. PURUSHOTTAM.

Solutions.

Question 1382.

(K. SATYANARAYANA):—Among the conics which have a given line for directrix and a given point and another given line for pole and polar, show that (1) there is only one parabola and (2) there are two central conics of given eccentricity whose centres are collinear with the given point

Analytical solution by G. V. Krishnaswami.

(1) Let the line $x = 0$ be the directrix, $(l, 0)$ the given point, $px + qy = r$ the given polar. Let (h, k) be the focus of a parabola satisfying the given conditions. The equation to the parabola is then

$$(y - k)^2 - 2xh + h^2 = 0.$$

Since $px + qy = r$ is the polar of $(l, 0)$ with respect to this parabola, we have

$$\frac{h}{p} = \frac{k}{q} = \frac{h^2 + k^2 - hl}{r} = t \text{ (say).}$$

$\therefore$ $$h = pt,\ k = qt, \text{ and } p^2t^2 + q^2t^2 - ptl = rt.$$

$\therefore$ $$t = \frac{pl + r}{p^2 + q^2}$$

rejecting the value $t = 0$ as not giving a proper parabola. Hence there is only one parabola.

(2) Under the same circumstances as in (1) the equation to a conic of eccentricity e is

$$x^2(1 - e^2) + y^2 - 2hx - 2ky + h^2 + k^2 = 0$$

Comparing as before $px + qy = r$ with the polar of $(l, 0)$ with respect to this conic we have

$$\frac{l(1 - e^2) - h}{p} = \frac{-k}{q} = \frac{hl - (h^2 + k^2)}{r} = t \text{ (say).}$$

$\therefore$ $$h = l(1 - e^2) - pt;\ k = -qt \qquad \ldots \text{(A)}$$

$\therefore$ $$rt = l^2(1 - e^2) - plt - [l(1 - e^2) - pt]^2 - q^2t^2,$$

a quadratic in t giving two values for (h, k). Hence there are *two* conics.

The centre of the conic is $\left(\frac{h}{1-e^2}, k\right)$. Hence in order that the centres may be collinear with $(l, 0)$ we should have

$$\begin{vmatrix} l, & 0, & 1 \\ h_1/(1-e^2), & k_1, & 1 \\ h_2/(1-e^2), & k_2, & 1 \end{vmatrix} = 0$$

i.e., $$l(k_1 - k_2) + \frac{1}{1-e^2}(h_1 k_2 - h_2 k_1) = 0 \qquad \ldots \text{(B)}$$

But from (A), by eliminating t, we have

$$h = l(1-e^2) + \frac{p}{q}k$$

i.e., $$h_1 = l(1-e^2) + \frac{p}{q}k_1$$

and $$h_2 = l(1-e^2) + \frac{p}{q}k_2.$$

By eliminating $\frac{p}{q}$ between these, we get relation (B).

Geometrical Solution by G. A. Srinivasan.

Let S be a focus, MX the corresponding directrix of a conic for which P and QR are pole and polar. If MX cuts QR at K and the diameter through P at M, it is well known that PSK is a right angle, and that SM is perpendicular to QR. Again, if the conic is central, and if PN be drawn perpendicular to the directrix, meeting SM at L, then

$$\text{PL} : \text{PN} = \text{CS} : \text{CX} = e^2,$$

where e is the eccentricity of the conic.

Thus, to find the focus of a conic of given eccentricity e, having a given line MX for its directrix and a point P and a line QR for pole and polar, we have the following construction:—

Draw PN perpendicular to MX and cut off PL along PN, equal to e^2 PN. Describe the circle on PK as diameter, where K is the point of intersection of MX and QR. Through L draw the perpendicular to QR cutting this circle at S and S′. Then S and S′ are the foci of the two conics satisfying the given conditions. If this line SS′ cuts the directrix at M, the centres of the two conics lie on MP.

If $e = 1$, L coincides with N, and the focus of the required parabola is the other point where the perpendicular from N on QR cuts the circle PNK.

Also solved by K. V. Vedantam.

Question 1429.

(S. D. CHOWLA) :—Prove that

$$\frac{1}{3^4} + \frac{3^4 - 1}{3^4 . 5^4} + \frac{(3^4 - 1)(5^4 - 1)}{3^4 . 5^4 . 7^4} + \frac{(3^4 - 1)(5^4 - 1)(7^4 - 1)}{3^4 . 5^4 . 7^4 . 11^4}$$

$$+ \frac{(3^4 - 1)(5^4 - 1)(7^4 - 1)(11^4 - 1)}{3^4 . 5^4 . 7^4 . 11^4 . 13^4} + \ldots = 1 - \frac{96}{\pi^2}$$

where 3, 5, 7, 11, 13, ... form the sequence of prime numbers.

Solution by S. Audinarayanan, M. V. Seshadri,
N. P. Subramanyan and T. Totadri Aiyangar.

Let P denote the series. Then, it is easily seen that

$$1 - P = \left(1 - \frac{1}{3^4}\right)\left(1 - \frac{1}{5^4}\right)\left(1 - \frac{1}{7^4}\right) \ldots \ldots$$

But $$\left(1 - \frac{1}{3^4}\right)^{-1}\left(1 - \frac{1}{5^4}\right)^{-1}\left(1 - \frac{1}{7^4}\right)^{-1} \ldots \ldots$$

$$= \frac{1}{1^4} + \frac{1}{3^4} + \ldots \ldots + \frac{1}{(2n-1)^4} + \ldots \ldots$$

$$= \frac{\pi^4}{96} \text{ (Hobson's } \textit{Plane Trigonometry}\text{, p. 368).}$$

$$\therefore \quad P = 1 - \frac{96}{\pi^4}.$$

Remarks by N. P. Subramanian.

In the note on "Some results involving prime numbers" pages *100—104*, Vol. XVI, No. 7, (February, 1926) of this *Journal*, S. D. Chowla has arrived at the following general result :—

$$\frac{1}{3^{2n}} + \frac{3^{2n} - 1}{3^{2n} . 5^{2n}} + \frac{(3^{2n} - 1)(5^{2n} - 1)}{3^{2n} . 5^{2n} . 7^{2n}} + \frac{(3^{2n} - 1)(5^{2n} - 1)(7^{2n} - 1)}{3^{2n} . 5^{2n} . 7^{2n} . 11^{2n}}$$

$$+ \ldots \ldots = 1 - \frac{2(2n)!}{(2^{2n} - 1) B_n \pi^{2n}}.$$

Question 1437.

(K. ANANDA RAO) :—If p is a prime number, prove that the necessary and sufficient condition that $p+2$ is also a prime number is

$$\frac{(p-1)!+1}{p} \equiv -\frac{p+3}{4} \pmod{p+2}$$

when p is of the form $(4m+1)$,

and
$$\frac{(p-1)!+1}{p} \equiv \frac{p+1}{4} \pmod{p+2}$$

when p is of the form $(4m+3)$.

Solution by S. D. Chowla, T. R. Raghavasastri, T. Totadri Aiyangar and T. Vijiaraghavan.*

First let p be a prime of the form $(4m+1)$ and k an integer such

that
$$0 \leqslant k \leqslant p+1,$$

and
$$\frac{(p-1)!+1}{p} \equiv -\frac{p+3}{4} + k \pmod{p+2}.$$

Then,
$$(p-1)! \equiv -1 - \frac{p(p+3)}{4} + pk, \qquad \pmod{p+2}$$

so that
$$(p+1)! \equiv p(p+1)\left[pk - \frac{p^2+3p+4}{4}\right] \qquad ,,$$

$$\equiv \{(p-1)(p+2)+2\}\left[pk - \frac{(p+1)(p+2)+2}{4}\right] \pmod{p+2}$$

$$\equiv 2pk - 1 \pmod{p+2}$$

since $p-1$ and $p+1$ are even, p being a prime.

Hence
$$(p+1)!+1 \equiv 2pk, \pmod{p+2}.$$

Hence, by Wilson's theorem, the necessary and sufficient condition that $+\ 2$ may also be a prime is that $k=0$.

The case where p is of the form $(4m+3)$ can be dealt with in an exactly similar manner.

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* *Vide* pp. 85—88 of Vol. XVII (No. 6, December 1927) of the *Journal* for a note on the general problem.

Question 1452.

(Selected):—ABCDEF is a hexagon inscribed in a conic S′ so that any two consecutive vertices are conjugate points with respect to another conic S; prove that

$$\theta^3 - 4\Delta(\theta\,\theta' - 2\,\Delta\,\Delta') = 0.$$

Solution by V. Ganapathi Aiyar.

Since any two consecutive vertices of ABCDEF are conjugate points with respect to S, it follows that ACE and BDF are polar triangles with respect to S. Let S″ be the polar reciprocal of S′ with respect to S. Then S″ will be inscribed in both the triangles ACE and BDF. Therefore S′ and S″ are such that there exist triangles inscribed in S′ and circumscribed to S″.

Taking the common self-conjugate triangle of S and S′ as the reference triangle, we may write their equations in the forms:

$$S \equiv x^2 + y^2 + z^2 = 0$$

$$S' \equiv ax^2 + by^2 + cz^2 = 0.$$

The equation of S″ will be

$$S'' \equiv bcx^2 + cay^2 + abz^2 = 0.$$

Let Δ_1, θ_1, θ_1', Δ_1' be the invariants of S′ and S″, so that

$$\Delta_1 = abc,\ \Delta_1' = a^2b^2c^2,\ \theta_1 = b^2c^2 + c^2a^2 + a^2b^2,\ \theta_1' = abc\,(a^2+b^2+c^2).$$

Applying the condition that a triangle can be inscribed in S′ and circumscribed to S″, we have

$$a^2b^2c^2\,(a^2+b^2+c^2)^2 = 4a^2b^2c^2\,(b^2c^2+c^2a^2+a^2b^2)$$

or
$$(a^2+b^2+c^2)^2 = 4\,(b^2c^2+c^2a^2+a^2b^2).$$

With the usual notation the invariants of S and S′ are

$$\Delta = 1,\ \Delta' = abc,\ \theta = a+b+c,\ \theta' = bc+ca+ab.$$

Substituting these values in the equation of condition,

$$\{\theta^2 - 2\theta'\}^2 = 4\{\theta'^2 - 2\Delta'\theta\}$$

or making this homogeneous by using $\Delta = 1$, we have

$$(\theta^2 - 2\theta'\Delta)^2 = 4\Delta^2\,(\theta'^2 - 2\Delta'\theta)$$

i.e.,
$$\theta^3 - 4\theta\theta'\Delta + 8\,\Delta^2\Delta' = 0.$$

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1497. (B. B. BAGI) :—

(1) Through the vertices of a triangle ABC, lines AP, BQ, CR are drawn equal and parallel to the sides EF, FD, DE of another triangle DEF. Show that the triangles ABC, PQR have the same centroid.

(2) Pairs of points P P′, Q Q′, R R′, are taken on the sides BC, CA, AB of a triangle ABC, such that PP′, QQ′, RR′ are proportional to BC, CA, AB. Show that the triangle PQR, P′Q′R′ have the same centroid.

Enunciate corresponding theorems for a polygon.

1498. (N. DURAIRAJAN) :—Show that for any point on the nine-point circle of a triangle the following relation holds:

$$\frac{\sin 2\theta_1}{r_1^2} + \frac{\sin 2\theta_2}{r_2^2} + \frac{\sin 2\theta_3}{r_3^2} = 0,$$

r_1, r_2, r_3 and $\theta_1, \theta_2, \theta_3$ being the tripolar and triangular co-ordinates of the point.

1499. (A. A. KRISHNASWAMI AYYANGAR) :—Show that for every triangle ABC in which I is the in-centre, there is another triangle A′B′C′ in which I′ is the ex-centre, and *vice versa*, such that

$$IA = I'A',\ IB = I'B',\ IC = I'C'.$$

If the two triangles be so placed that I and I′ co-incide, show that the in-circle of the triangle ABC touches the circum-circle of the triangle A′B′C′, and the ex-circle of the triangle A′B′C′ touches the circum-circle of the triangle ABC; *if*, in addition, the circum-centres of the triangles ABC, A′B′C′ are in a line with I or I′ (which have already been made to coincide) then the circum-circles of the two triangles also touch each other.

1500. (S. NARAYANA AIYAR, M.A.) :—Establish the following relations :—

$$(1)\quad \text{dn}\,\frac{K}{3} = k'\,.\,\frac{1 + k^2\,\text{sn}^2\,\frac{K}{3}\,\text{sn}^2\,\frac{2K}{3}}{1 - k^2\,\text{sn}^2\,\frac{K}{3}\,\text{sn}^2\,\frac{2K}{3}};$$

$$(2)\quad \text{dn}\,\frac{K}{2} = k'\,.\,\frac{1 + k^2\,\text{sn}^2\,\frac{K}{4}\,\text{sn}^2\,\frac{3K}{4}}{1 - k^2\,\text{sn}^2\,\frac{K}{4}\,\text{sn}^2\,\frac{3K}{4}};$$

$$(3)\quad \text{dn}\,\frac{3K}{5} = k'\,.\,\frac{1 + k^2\,\text{sn}^2\,\frac{K}{5}\,\text{sn}^2\,\frac{4K}{5}}{1 - k^2\,\text{sn}^2\,\frac{K}{5}\,\text{sn}^2\,\frac{4K}{5}}.$$

1501. (K. J. SANJANA, M.A.) :—

(i) Prove that one solution of the equation

$$x^2 - 67y^2 = 1 \text{ is } x = 48842;\ y = 5967,$$

and show that successive values of x and y can be found to satisfy the equation.

(ii) Prove that one solution of the equation

$$x^2 - 61y^2 = 1 \text{ is } x = 1766319049;\ y = 226153980,$$

and show how other solutions may be obtained.

[These solutions have been taken from the *Kuttaka*, of Bháskara's *Bíja Ganita*. I give below the *sloka* in which he has propounded the question :—

Ká saptashashthiguṇitá kritirekayuktá,
Ká chaikashashthinihatá cha sukhe sarúpá |
Syánmúladá yadi kritiprakritir-nitántam
Tvachchetasi pravada táta tatá latávat ||].

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LIST OF JOURNALS AND PERIODICALS RECEIVED IN THE LIBRARY

during the months of January and February 1928.

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1 Acta Mathematica, **50**, 3 and 4, **51**, 1 and 2.
2 American Journal of Mathematics, **49**, 4.
3 American Mathematical Monthly, **34**, 10.
4 Annales de l'ecole Normale Superieure, May — Sept. 1927.
5 Astrophysical Journal, **56**, 3.
6 Bulletin of the American Mathematical Society, **34**, 1.
7 Bulletin of the Calcutta Mathematical Society, **18**, 4. (2 copies).
8 Bulletin des Sciences Mathematiques, Nov. 1927.
9 Crelle's Journal, **158**, 3 and 4.
10 Japanese Journal of Mathematics, **4**, 3.
11 Mathematical Gazette, **13**, 191, **14**, 192. (3 copies each).
12 Messenger of Mathematics, **57**, 3. (2 copies).
13 Monthly Notices of the Royal Astronomical Society, **88**, 1 and 2.
14 Philosophical Magazine, **5**, 27.
15 Popular Astronomy, **36**, 1.
16 Proceedings of the London Mathematical Society, **27**, 1—4, (2 copies).
17 Proceedings of the Royal Society, London, **117**, 776 and 777.
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